

Algorithms & Data Structures

Exercise sheet 12

HS 25

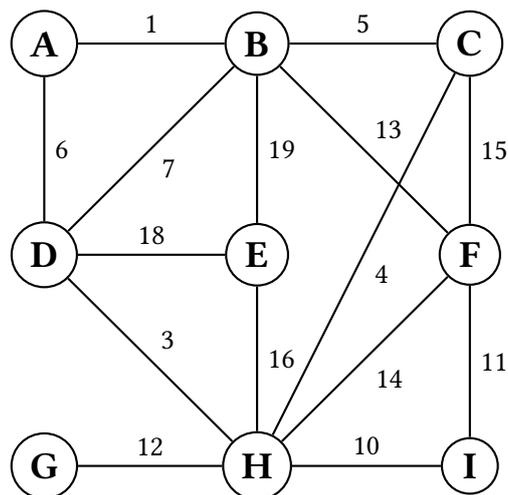
The solutions for this sheet are submitted on Moodle until 14 December 2025, 23:59.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

Exercise 12.1 *MST practice (1 point).*

Consider the following graph.



- Compute the minimum spanning tree (MST) using Boruvka's algorithm. For each step, provide the set of edges that are added to the MST.
- Provide the order in which Kruskal's algorithm adds the edges to the MST.
- Provide the order in which Prim's algorithm (starting at vertex G) adds the edges to the MST.

Exercise 12.2 *Constructing a Fiber Optic Network.*

The government of Atlantis put you in charge of installing a fiber optic network that connects all its n cities. There are two technologies of fibre optic that you can use:

- Fibre 1.0: It is a good reliable technology that is relatively cheap. There is a list of pairs of cities between which it is possible to install a direct Fibre 1.0 link. Furthermore, for each such pair, there is a corresponding positive integer cost.
- Fibre 2.0: It is an emerging technology that is extremely good and can directly connect any two cities. However, its cost is too high and the government cannot afford a single Fibre 2.0 link.

Note that all direct links are two-directional. The installed network should connect all the cities of Atlantis: Between any two cities, there should be a connected path of direct links in the network that connects them.

A philanthropist volunteered to donate the cost of exactly k direct Fibre 2.0 links ($k < n$), and you can use them to connect any k pairs of cities. Your goal is to minimize the cost that is paid by the government for the Fibre 1.0 links that are needed to construct a connected network. Describe an algorithm that finds the network that costs the government the minimum amount of money.

Note that it is possible to construct a network connecting all the cities of Atlantis using only Fibre 1.0 links, but we would like to benefit from the k Fibre 2.0 links that were donated by the philanthropist in order to minimize the cost that is paid by the government.

Hint: Modify Kruskal's algorithm.

Exercise 12.3 *Uniqueness of MSTs (1 point).*

The goal of this exercise is to understand when a graph has a unique minimum spanning tree.

- (a) Give an example of a graph for which the minimum spanning tree is not unique. Show how to get two different minimum spanning trees of this graph using Kruskal's or Prim's algorithm. When there is a choice because several edges have the same weight, the algorithms are allowed to pick any of those edges.

It turns out that for a connected graph, if the weights of the edges are pairwise distinct, the minimum spanning tree is unique. To show this, let $G = (V, E)$ be a connected graph and $w : E \rightarrow \mathbb{R}_{\geq 0}$ be a weight function such that $w(e) \neq w(f)$ for $e, f \in E$ with $e \neq f$. We assume by contradiction that there are two different minimum spanning trees T_1 and T_2 . Out of all edges that are in $T_1 \setminus T_2$ or $T_2 \setminus T_1$, let e be the edge of minimum weight (the edge of minimum weight is unique since by assumption the edge weights are pairwise distinct). By exchanging the roles of T_1 and T_2 if necessary, we can assume that $e \in T_1 \setminus T_2$.

- (b) Show that $T_2 \cup \{e\}$ has a cycle and that there is an edge $f \in T_2 \setminus T_1$ on this cycle that has strictly larger weight than e .
- (c) Show that the minimum spanning tree of G with the weight function w is unique.

Hint: Use part (b) to construct a spanning tree of smaller weight than T_2 .

- (d) Is the converse true as well? That is, if $G = (V, E)$ is a connected graph that has a unique minimum spanning tree, are the edge weights necessarily distinct? Give a proof or counterexample.

Exercise 12.4 *Counting Minimum Spanning Trees With Identical Edge Weights (1 point).*

Let $G = (V, E)$ be an undirected, weighted graph with weight function w .

As we saw in Exercise 12.3, if G is connected and all its edge weights are pairwise distinct¹, then its Minimum Spanning Tree is unique. You can use this fact without proof in the rest of this exercise.

An edge is called redundant if there exists another edge with the same weight. A graph is k -redundant if it has exactly k redundant edges. More formally this means that

$$|\{e \in E \mid \exists e' \in E. e \neq e' \wedge w(e) = w(e')\}| = k.$$

¹I.e., for all $e \neq e' \in E$, $w(e) \neq w(e')$.

In particular, if G 's edge weights are all distinct, then G is 0-redundant, and if its edge weights are all identical, it is $|E|$ -redundant.

(a) Given a weighted graph $G = (V, E)$ with weight function c and $e = \{v, w\} \in E$, we say that we *contract* e when we perform the following operations:

(i) Replace v and w by a single vertex vw in V , i.e., $V' \leftarrow V - \{v, w\} \cup \{vw\}$.

(ii) Replace any edge $\{v, x\}$ or $\{w, x\}$ by an edge $\{vw, x\}$ in E , i.e.,

$$E' \leftarrow E - \{\{v, x\} \mid x \in V\} - \{\{w, x\} \mid x \in V\} \cup \{\{vw, x\} \mid \{v, x\} \in E \vee \{w, x\} \in E\}.$$

(iii) Set the weight of the new edges to the weight of the original edges, taking the minimum of the two weights if two edges are merged, i.e.

$$\begin{aligned} c'(\{x, y\}) &= c(\{x, y\}) & x, y \notin \{v, w\} \\ c'(\{vw, x\}) &= c(\{v, x\}) & \{v, x\} \in E, \{w, x\} \notin E \\ c'(\{vw, x\}) &= c(\{w, x\}) & \{v, x\} \notin E, \{w, x\} \in E \\ c'(\{vw, x\}) &= \min(c(\{v, x\}), c(\{w, x\})) & \{v, x\} \in E, \{w, x\} \in E. \end{aligned}$$

For all $G = (V, E)$ and $e \in E$, we denote by G_e the graph obtained by contracting e in G . Explain why if T is an MST of G and $e \in T$, then T_e must be an MST of G_e .

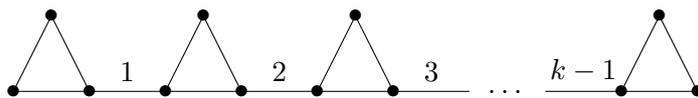
(b) Let $k > 0$. Show that for all k -redundant $G = (V, E)$ and $e \neq e' \in E$ with $w(e) = w(e')$, then G_e is k' -redundant for some $k' \leq k - 1$.

(c) Show that if G is connected and k -redundant, it has at most 2^k distinct MSTs.

Hint: By induction over k , using (a) and (b).

(d)* Show that for all large enough n , there exists a graph G such that G is n -redundant and has at least $2^{\frac{n}{2}}$ distinct MSTs.

Hint: First assume that $n = 3k$ for some k . Consider graphs of the following form, where all unmarked edges have weight 0. When $n = 3k + 1$ or $n = 3k + 2$, you can add one or two edges with cost k and $k + 1$ at either end.



Exercise 12.5 Heavy and light edges.

Let $G = (V, E)$ be a connected, undirected, weighted graph with positive weights $w_e > 0$ for $e \in E$. We say an edge $e \in E$ is *heavy* if there exists a cycle $C \subseteq E$ so that $e \in C$ is the (strictly) heaviest edge in C , i.e.,

$$w_e > w_f \text{ for all } f \in C \text{ with } f \neq e.$$

We say an edge is *light* if there exists a minimum spanning tree $T \subseteq E$ of G which contains e .

(a) Show that a heavy edge cannot be light.

Hint: Assume for a contradiction that $T \subseteq E$ is an MST of G and that T contains a heavy edge e . Say e is the heaviest edge in a cycle $C \subseteq E$. Construct a strictly cheaper spanning tree of G by removing e from T , and replacing it by a different edge $f \in C$.

(b*) Show that an edge which is not heavy, must be light. Conclude that an edge is heavy if and only if it is not light.

Hint: You may use without proof that Kruskal's algorithm is correct regardless of the order in which edges of equal weight are processed.