Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 3

This is the exercise sheet number 3. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercises are Exercise 3.2 and 3.7 and your solution has to be uploaded on the Moodle page of the course **by 09/10/2025, 23:59**. The solution to these exercises must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

1 Predicate Logic

Exercise 3.1 Expressing Relationship of Humans in Predicate Logic ($\star$)

Consider, as in the lecture, the universe of all humans (including those who died) and the following predicate:

$$\texttt{par}(x, y) = 1 \iff \text{“}x\text{ is parent of }y.\text{”}$$

Express the following statements as a formula in predicate logic, using only the above predicates (in particular, do **not** use the predicate `equals`, often also written as $=$).

1. x is great-grandparent of y.
2. x and y are (first) cousins.

Exercise 3.2 From Natural Language to a Formula ($\star$) — GRADED *(4 points)*

Please upload your solution by 09/10/2025

Consider the universe $U = \mathbb{N} \setminus \{0\}$. Express each of the following statements with a formula in predicate logic, in which the only predicates appearing are $\texttt{smallerthan}(x, y)$, $\texttt{divides}(x, y)$, $\texttt{equals}(x, y)$ and $\texttt{prime}(x)$ (instead of $\texttt{smallerthan}(x, y)$, $\texttt{divides}(x, y)$ and $\texttt{equals}(x, y)$ you can write $x < y$, $x \mid y$ and $x = y$ accordingly). You can also use the symbols $+$ and $\cdot$ to denote the addition and multiplication functions, and you can use constants (e.g., $0, 1, \ldots$). You can also use $\longrightarrow$ and $\longleftrightarrow$. No justification is required.

1. ($\star$) There does not exist a largest natural number.
2. ($\star$) The only divisors of a prime number are 1 and the number itself.
3. ($\star$) 1 is the only natural number which has an inverse.
4. ($\star$) A prime number divides the product of two natural numbers if and only if it divides at least one of them.

Exercise 3.3 Winning Strategy ($\star\star$)

Alice and Bob play a game in which the stake is a chocolate bar. Rules of the game are the following: Alice chooses two integers a_1, a_2 and Bob chooses two integers b_1, b_2. Alice wins whenever $a_1 + (a_2 + b_1)^{|b_2|+1} = 1$ and Bob wins otherwise.

1. First, consider the case when Alice and Bob announce all their numbers at the same time. Give a formula that describes the statement "Alice has a winning strategy." Is this statement true?

2. In the second case, Alice and Bob announce their numbers one by one. That is, first Alice announces a_1, then Bob announces b_1, then Alice announces a_2, and at the end Bob replies with b_2. Once again, give a formula that describes the statement "Alice has a winning strategy." Is this statement true in this case?

2 Proof Patterns

Exercise 3.4 Indirect Proof of an Implication

Prove indirectly that for all natural numbers $n > 0$, we have:

1. ($\star$) If n^2 is odd, then n is also odd.

2. ($\star\star$) If $42^n - 1$ is a prime, then n is odd.

Exercise 3.5 Case Distinction

Prove by case distinction that:

1. ($\star$) $n^3 + 2n + 6$ is divisible by 3 for all natural numbers $n \geq 0$.

2. ($\star\star$) If p and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.

Exercise 3.6 Proof by Contradiction

1. ($\star\star$) Show by contradiction that the sum of a rational number and an irrational number is irrational.

   Hint: Use the fact that the difference of two rational numbers is rational.

2. ($\star\star\star$) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

   Hint: Fermat's Last Theorem states that no positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for $n > 2$.

Exercise 3.7 New Proof Patterns ($\star$) — GRADED *(4 points)*
Please upload your solution by 09/10/2025

For each of the following proof patterns, **prove** or **disprove** that it is sound. Do so by first writing it as a statement involving logical consequence on formulas and then proving that the resulting statement is either true or false.

1. ($\star$) To prove a statement S, find two appropriate statements T_1 and T_2. Assume that S is false and show (from this assumption) that at least one of the statements T_1 and T_2 is true. Then show that at least one of the statements T_1 and T_2 is false.

2. ($\star$) To prove an implication $S \Rightarrow T$, find an appropriate statement R. First, show that R is false. Then, assume that S is true and T is false, and prove that (from these assumptions) R is true.

Due by 09/10/2025, 23:59.
Exercise 3.2 and 3.7 will be graded.