Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

Marian DIETZ — Milan GONZALEZ-THAUVIN — Zoé REINKE

Exercise sheet 4

This is the exercise sheet number 4. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 4.5 and your solution has to be uploaded on the Moodle page of the course **by 16/10/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 4.1 Pigeon set ($\star\star$)

Let n be an integer such that $n \geq 3$, and let S be the set $S = \{1, 2, \cdots, n-1\}$.
Prove that every subset of S of size $\lfloor \frac{n}{2} \rfloor + 1$ contains two elements whose sum is n.
The floor function $x \mapsto \lfloor x \rfloor$ is the function which takes a real number x as argument and outputs the greatest integer smaller or equal to x.
Hint 1: Try with small values of n.
Hint 2: Use case distinction and the pigeonhole principle.

Exercise 4.2 Element or Subset ($\star$)

For each of the following choices of sets A and B, decide which of the statements $A \in B$ and $A \subseteq B$ are true.

1. (a) $A = \{1, 0, \{0\}, 1\}, B = \{\{0, 1, 0, \{0, 0\}\}, 1, 10, 0\}$
   (b) $A = \emptyset, B = \{\{\emptyset\}, \{\emptyset, \emptyset\}, \emptyset\}$
   (c) $A = \{\{0\}, 0, \{0\}, \{\{0\}\}\}, B = \{0, \emptyset, \{\{0\}\}, \{0\}\}$
   (d) $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset, \emptyset, \emptyset\}\}$

Exercise 4.3 Operations on Sets ($\star$)

In each of the following cases, give a set A such that

1. (a) There exists an $x \in A$ such that $x \subseteq A$.
   (b) $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$.
   (c) $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

Exercise 4.4 Cardinality ($\star$)

Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset\}\}$ and $B = \{A, \{\emptyset\}, \{\{\emptyset\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.

1. $A \cup B$
2. $A \cap B$
3. $\emptyset \times A$
4. $\{0\} \times \{3, 1\}$
5. $\{\{1, 2\}\} \times \{3\}$
6. $\mathcal{P}(\{\emptyset\})$

Exercise 4.5 Symmetric difference ($\star\star$) — GRADED *(8 points)*
Please upload your solution by 16/10/2025

In this exercise, we introduce a new operator, the symmetric difference. The symmetric difference of sets A and B, denoted $A\Delta B$, is the set of elements belonging to A or B but not to both:

$$A\Delta B \stackrel{\text{def}}{=} \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\}$$

1. Rewrite the definition above using set operations. More specifically, give an expression using only "A", "B", "$\cap$", "$\cup$", and "$\setminus$" (and parentheses), which is equal to the set $A\Delta B$. **Each of the operators "$\cap$", "$\cup$", and "$\setminus$" may appear at most once**. No justification is required.

2. Prove that for all sets A and B, it holds that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.
Expectation: You should give a full proof for the aforementioned statement using set operations and/or the tools from previous chapters (rules of Lemma 2.1, function table, logical consequences, proof patterns). Formalism is important, but in contrast to the bonus exercise on sheet 2, you don't have to use rules of Lemma 2.1 one by one. You may use many rules at once, as soon as there is a main one (distributivity without specifying which, de Morgan's rule, absorption) used as justification and an unlimited number of minor rules (associativity, commutativity, double negation, idempotence) not necessarily mentioned. But there must be a justification for each step, even if the step uses solely a minor rule. Also, the **same** main rule can be used many times in one step. You may also use the following equivalences: $F \wedge \top \equiv F$, $F \wedge \bot \equiv \bot$, $F \vee \bot \equiv F$ and $F \vee \top \equiv \top$. **Recall that every step of your proof should be justified.**

3. Prove that for all sets A, B, and C, the following holds:

$$A\Delta B = A\Delta C \implies B = C\ .$$

Expectation: Same as above.

Exercise 4.6 Relating Two Power Sets ($\star\star$)

Prove or disprove each of the following statements.

1. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B.
2. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for any sets A and B.
3. $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B.

Exercise 4.7 Special Families of Sets ($\star\star$)

Let $X \neq \emptyset$ be a set. We define the following predicate:[1]

$$Q_X(\mathcal{A}) = 1 \iff \begin{cases} \mathcal{A} \subseteq \mathcal{P}(X), \\ \mathcal{A} \neq \emptyset, \\ A \cup B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ A \cap B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ X \setminus A \in \mathcal{A} \text{ for all } A \in \mathcal{A}. \end{cases}$$

Prove or disprove the following statements.

1. $Q_X(\mathcal{P}(X)) = 1$.
2. $Q_X(\{X\}) = 1$.
3. For all $\mathcal{A} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ then $X \in \mathcal{A}$.
4. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cup \mathcal{B}) = 1$.
5. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cap \mathcal{B}) = 1$.

Exercise 4.8 Short questions (exam 2022) ($\star$)

This exercise is taken from the end of semester exam of 2022. We decided to keep the exam grading scale but this exercise is not a (graded) bonus exercise for this week. Each correct answer gives one point. No justification is required.

1. How many elements does the set $\{\{0,1\}, \{0\} \times \{1\}\} \times \{0\}$ have?
2. List each element of the set $\{\varnothing, (0,1)\} \times \{\{0\} \cup \{1\}, \{1,0\}\}$ exactly once.
3. Find sets A, B, and C such that $A \setminus B \subset A \setminus C$.
4. Find a set A such that $A \cap \mathcal{P}(A) \neq \varnothing$.

[1] This notation stand for the logical *conjunction* of all statements on the right, meaning the predicate is true if and only if *all* statements on the right are true.

Exercise 4.9 Short questions (exam 2021) ($\star$)

This exercise is taken from the end of semester exam of 2021. We decided to keep the exam grading scale but this exercise is not a (graded) bonus exercise for this week.
Each correct answer gives the indicated number of points. An unanswered question gives zero points. For each wrong answer, the indicated number of points are deducted. Overall, at least 0 points are given for the whole task. No justification is required.

1. (1 point) True or False: the sets $\varnothing \times \varnothing$ and $\{\varnothing\} \times \{\varnothing\}$ are equal.
2. (1 point) True or False: the set $\{\varnothing, \{\varnothing\}\}$ is a subset of $\{\varnothing\}$.
3. (1 point) True or False: for any two finite sets A and B we have $|A \cup B| = |A| + |B|$.
4. (1 point) True or False: for any two sets A and B there exists a set C such that

$$A = B \cap C \quad \text{or} \quad B = A \cap C.$$

5. (1 point) True or False: for any set A, we have $A \cap \mathcal{P}(A) = \varnothing$.

Due by 16/10/2025, 23:59.
Exercise 4.5 will be graded.