Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 5

This is the exercise sheet number 5. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 5.4 and your solution has to be uploaded on the Moodle page of the course **by 23/10/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 5.1 Computing Representations of Relations ($\star$)

For the relation $\rho = \{(1,4),(2,1),(2,3),(4,2)\}$ on the set $\{1,2,3,4\}$, determine the relations ρ^3 and ρ^*. Describe ρ^3 using the set representation, and ρ^* using matrix representation (see lecture notes, Section 3.3.2).

Exercise 5.2 Operations on Relations ($\star\star$)

Let us consider the relations $<, |$ and $\equiv_2$ on the set of positive natural numbers $\mathbb{N} \setminus \{0\}$. For each of the following relations on $\mathbb{N} \setminus \{0\}$, decide whether it is reflexive, symmetric or transitive. Justify your answers.

1. $< \circ \mid$
2. $\mid \cup \equiv_2$
3. $\mid \cup \widehat{\mid}$

Exercise 5.3 An Equivalence Relation ($\star\star$)

The relation $\sim$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is defined as follows:

$$(x_1, y_1) \sim (x_2, y_2) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \exists \lambda > 0 \; (x_1, y_1) = (\lambda x_2, \lambda y_2)$$

1. Prove that $\sim$ is an equivalence relation.
2. Describe geometrically the equivalence classes $[(x, y)]$.

Exercise 5.4 Properties of Relations ($\star$) — GRADED *(8 points)*
Please upload your solution by 23/10/2025

Prove or disprove the following claims:

1. Let ρ be a relation on a set A. Then, the relation $\gamma \stackrel{\text{def}}{=} \rho \circ \widehat{\rho}$ is symmetric.
2. Let A be a set and let σ be a symmetric relation on A and π be an antisymmetric relation on A. Then, the relation $\gamma \stackrel{\text{def}}{=} \sigma \circ \widehat{\pi}$ is symmetric.
3. The intersection of two equivalence relations on the same set is an equivalence relation.

Exercise 5.5 Lifting an Operation to Equivalence Classes ($\star\star$)

In the lecture (and in Section 3.4.3 of the lecture notes), we have considered the following equivalence relation $\sim$ on the set $A \stackrel{\text{def}}{=} \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$:

$$(a, b) \sim (c, d) \stackrel{\text{def}}{\iff} ad = bc.$$

We then defined the rational numbers $\mathbb{Q} \stackrel{\text{def}}{=} A/\!\sim$, capturing the fact that each rational number has different **representations** as fractions. The goal of this exercise is to understand how to define an operation on equivalence classes (e.g., the sum of two rational numbers) by lifting an operation defined at the level of the **elements** of the equivalence classes (e.g., the fractional representations A of the rationals). Consider an equivalence relation θ on some set B and a function $f : B^2 \to B$. We want to lift f to the set of equivalence classes B/θ, i.e., we want to define a function $F : (B/\theta)^2 \to (B/\theta)$ canonically in terms of f. For this to be meaningful, f has to be **θ-consistent**. That is, if f is applied to a pair $(b_1, b_2) \in B^2$, the equivalence class $[f(b_1, b_2)]_\theta$ may only depend on the **equivalence classes** $[b_1]_\theta$ and $[b_2]_\theta$ (irrespective of which concrete elements both b_1 and b_2 are within their equivalence classes).

1. Define the sum of two fractional representations of rational numbers as a function $\mathsf{sum} : A^2 \to A$ (for $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as defined above), using standard addition and multiplication in the integers.
2. Express formally what it means for a function $f : B^2 \to B$ to be θ-consistent for an equivalence relation θ on B.

   Hint: For example, the function sum defined in question 1 being $\sim$-consistent captures the fact that when adding two rational numbers x and y, we can add two **arbitrary** fractional representations of x and y from the set A (by applying sum), and the rational number obtained does not depend on which representation of x and y was used.
3. Prove that the function $\mathsf{sum} : A^2 \to A$ defined in question 1 is $\sim$-consistent.

Exercise 5.6 Antisymmetry (exam 2021) ($\star$)

This exercise is taken from the end of semester exam of 2021.
Let ρ and σ be relations on some non-empty set A. Prove or disprove:
If ρ is antisymmetric and σ is antisymmetric, then $\rho \circ \sigma$ is antisymmetric.

Due by 23/10/2025, 23:59.
Exercise 5.4 will be graded.