Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 6

This is the exercise sheet number 6. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 6.5 and your solution has to be uploaded on the Moodle page of the course **by 30/10/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 6.1 Partial Order Relations ($\star$)

1. Consider the poset $(\mathbb{N} \setminus \{0\};\ |\ )$. Which of the following pairs are comparable?

   (i) $(11, 12)$ (ii) $(4, 6)$ (iii) $(5, 15)$ (iv) $(42, 42)$

2. Consider the set $A \stackrel{\text{def}}{=} (\mathbb{N} \setminus \{0\}) \times (\mathbb{N} \setminus \{0\})$ with the lexicographic order $\leq_{\mathsf{lex}}$ defined as:

$$(a, b) \leq_{\mathsf{lex}} (c, d) \stackrel{\text{def}}{\Longleftrightarrow} (a \neq c \wedge a \mid c) \vee (a = c \wedge b \mid d).$$

   Determine all elements $a \in A$, such that $a \leq_{\mathsf{lex}} (2, 5)$.

3. Prove or disprove: If $(A; \preceq)$ is a poset, then $(A; \widehat{\preceq})$ is also a poset.

Exercise 6.2 Hasse Diagrams ($\star$)

For each of the two posets: $(\{1, 2, 3\}; \leq)$ and $(\{1, 2, 3, 5, 6, 9\};\ |\ )$, draw the Hasse diagram and determine all least, greatest, minimal and maximal elements.

Exercise 6.3 The Lexicographic Order ($\star\star$)

Prove Theorem 3.13 from the lecture notes:
For given posets $(A; \preceq)$ and $(B; \sqsubseteq)$, the relation $\leq_{\mathsf{lex}}$ defined on $A \times B$ by

$$(a_1, b_1) \leq_{\mathsf{lex}} (a_2, b_2) \stackrel{\text{def}}{\Longleftrightarrow} a_1 \prec a_2 \vee (a_1 = a_2 \wedge b_1 \sqsubseteq b_2)$$

is a partial order relation.
NB: $a_1 \prec a_2$ means $a_1 \preceq a_2 \wedge a_1 \neq a_2$.

Exercise 6.4 Inverses of Functions ($\star\star$)

For a set A, the identity function id_A is the function $\mathrm{id}_A : A \to A$ that maps any element of A to itself. Consider a function $f : A \to A$. Prove that there exists a function $g : A \to A$ such that $g \circ f = \mathrm{id}$ if and only if f is injective. Such g is called a **left inverse** of f.

Exercise 6.5 Equinumerous sets — GRADED *(8 points)*
Please upload your solution by 30/10/2025

Prove the following statements.

1. For all sets A, A and $\mathcal{P}(A)$ are <u>not</u> equinumerous.
   Hint: You could consider the set $B = \{x \in A \mid x \notin f(x)\}$ for a well-chosen function f.

2. The sets $C \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x \leq 1\}$ and $D \stackrel{\text{def}}{=} \{x \in \mathbb{Q} \mid 0 \leq x < 1\}$ are equinumerous. You can use any lemma, theorem or corollary of Section 3.7 of the lecture notes in your proof but you must mention it.

Exercise 6.6 Uncountability

Prove that the following sets are uncountable.

1. ($\star$) The set A of all semi-infinite sequences over $\{0, 1, \ldots, 9\}$.

2. ($\star$) The set C of all points on the unit circle, that is $C \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

3. ($\star\,\star$) for all $\ell \in \mathbb{N}$ with $\ell \geq 1$ the set

$$A_\ell := \left\{ f \in \{0,1\}^\infty \;\middle|\; \sum_{i=0}^{k} f(i) \leq \frac{k}{\ell} + 1 \text{ for all } k \in \mathbb{N} \right\}.$$

   Hint: Recall that an infinite bit sequence $f \in \{0,1\}^\infty$ can be defined formally as a function $f : \mathbb{N} \to \{0, 1\}$.
   Hint: For all $\ell \geq 1$, explicitly write an injection from a known uncountable set into A_ℓ.

4. ($\star\,\star\,\star$) The set

$$S = \big\{ f : \mathbb{N} \to \{0,1\} \,\big|\, \forall n, m \in \mathbb{N} : f(n) = 0 \land n \mid m \to f(m) = 0 \big\}.$$

   Hint: you can use the fact that there are infinitely many prime numbers.

Exercise 6.7 fogof (exam 2022) ($\star\,\star$)

This exercise is taken from the end of semester exam of 2022.
Let X and Y be two non-empty sets. Moreover, let $f : X \to Y$ and $g : Y \to X$ be functions such that $f \circ g \circ f$ is injective and surjective. Prove the following statements:

1. f is injective.

2. f is surjective.

3. g is injective.

4. g is surjective.

Due by 30/10/2025, 23:59.
Exercise 6.5 will be graded.