Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 8

This is the exercise sheet number 8. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 8.4 and your solution has to be uploaded on the Moodle page of the course **by 13/11/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 8.1 Algebras ($\star$)

For each of the following algebras, decide whether it is a monoid, a group or neither. In case it is a monoid or a group, decide whether it is commutative. Justify your answers.

1. $\langle \mathbb{Z}; \star \rangle$, where $\star$ is defined by $a \star b \stackrel{\text{def}}{=} a^2 + b^2$ for any $a, b \in \mathbb{Z}$.
2. $\langle \mathcal{P}(X); \cup \rangle$, where X is a non-empty finite set.
3. $\langle S; * \rangle$, where $S = (\mathbb{Q} \setminus \{0\}) \times \mathbb{Q}$ and

$$(a, b) * (c, d) \stackrel{\text{def}}{=} (ac, ad + b).$$

Exercise 8.2 Facts About Groups ($\star\star$)

In this exercise you are **not** allowed to use lemmas from the lecture notes (especially, Lemma 5.3). Let $\langle G; *, \widehat{\ }, e \rangle$ be a group.

1. Prove that the group axiom **G2** can be simplified (see also Section 5.2.4 of the lecture notes). That is, show that **G2** follows from the axioms **G1**, **G2′** and **G3**, where

   G2′ e is a right neutral element: $a * e = a$ for all $a \in G$.
2. Prove that $\widehat{a * b} = \widehat{b} * \widehat{a}$ for all $a, b \in G$.
3. Prove that $a * b = a * c \implies b = c$ for all $a, b, c \in G$.

Exercise 8.3 Group Structure Induced by Bijections ($\star\star$)

Let $\langle G, *, \widehat{\ }, e \rangle$ be a group, and let S be a set. Assume that $f : G \to S$ is a bijection, and consider the binary operation $\star$ on S given by $s \star s' \stackrel{\text{def}}{=} f\left(f^{-1}(s) * f^{-1}(s')\right)$ as well as the unary operation $\widetilde{\ }$ on S given by $\widetilde{s} \stackrel{\text{def}}{=} f\left(\widehat{f^{-1}(s)}\right)$. Prove the following statements.

1. Axiom **G1** holds for $\langle S, \star, \widetilde{\ }, f(e) \rangle$.

2. Axioms **G2** and **G3** hold for $\langle S, \star, \tilde{}, f(e)\rangle$.

3. $f : G \to S$ is a group isomorphism.

4. For all non-empty countable sets A, there exists a group with A as carrier.

Exercise 8.4 Equivalence relation from a subgroup ($\star\star$) — GRADED *(8 points)*
Please upload your solution by 13/11/2025

Let $\langle G; \cdot, \widehat{}, e\rangle$ be an abelian (= commutative) group and let H be a subgroup of G. We define a relation $\sim$ over G the following way[1]:

$$\forall a, b \in G,\ a \sim b \overset{\text{def}}{\iff} \exists h \in H \text{ such that } a \cdot h = b$$

1. Prove that $\sim$ is an equivalence relation.

Denote by G/H the set of all equivalence classes of the relation $\sim$ (which is a partition of G, see Theorem 3.11). Let $\pi : G \to G/H$ be the function defined by $\pi(x) \overset{\text{def}}{=} [x]_\sim$, where $[a]_\sim$ denotes the equivalence class of an element $a \in G$ for the relation $\sim$.

2. Let $a, a', b, b' \in G$ such that $a \sim a'$ and $b \sim b'$. Show that $\pi(a \cdot b) = \pi(a' \cdot b')$.

Thanks to the previous question, it is possible to define without ambiguity the binary operation $\star$ on G/H as

$$\pi(a) \star \pi(b) \mapsto \pi(a \cdot b)$$

3. Show that $\langle G/H; \star\rangle$ is a group.
   Expectations: In this question, you have to
   - Prove that $\star$ is associative.
   - Specify an element $e' \in G/H$ and prove that it is neutral in the sense of Definition 5.7.
   - Provide an inverse operation $\widehat{\ }$, **prove that it is well defined (if necessary)** and that every element of G/H has an inverse.

4. Is π a homomorphism? No justification is needed.

[1] A small note on formalism: in the set chapter we defined $\in$ as as the binary predicate: $a \in X$ if and only if a belongs to a set X. When many elements belong to the same set, a correct notation should therefore be $(a, b) \in X^2$, but for the sake of simplicity we sometimes write "$a, b \in X$" instead of "$a \in X$ and $b \in X$". This can be seen as syntactic sugar and not as a mathematical extension of $\in$. For instance, "$a, b \notin X$" is unclear and should not be used, and "$(a, b) \in X$" is absolutely wrong (in that context).

Exercise 8.5 A Binary Operation From a Group Homomorphism ($\star\star$)

Let $\langle G; *, \widehat{\ }, e\rangle$ be a group, and let $\psi : G \to G$ be a group homomorphism. Consider the algebra $\langle G; \cdot\rangle$ with

$$x \cdot y \stackrel{\text{def}}{=} \psi(x) * \psi(y) \quad \text{for any } x \in G \text{ and } y \in G.$$

We say that a function $f : X \to X$ is **idempotent** if $f(x) = f(f(x))$ for all $x \in X$.
Prove that

The binary operation $\cdot$ is associative if and only if ψ is idempotent.

Exercise 8.6 Isomorphisms Map Generators to Generators ($\star\star$)

Let ψ be a group isomorphism from $\langle G; *, \widehat{\ }, e\rangle$ to $\langle H; \star, \widetilde{\ }, e'\rangle$. Prove that if G is cyclic and g is a generator of G then $\psi(g)$ is a generator of H.

Exercise 8.7 Kernel of an homomorphism (exam 2024) ($\star$)

This exercise is taken from the winter exam of 2024.
Consider groups $\langle G; +, \widehat{\ }, e_G\rangle$ and $\langle H; \odot, \widetilde{\ }, e_H\rangle$. Let $\phi : G \to H$ be a group homomorphism. Consider the subset of G defined as:

$$\ker(\phi) = \{g \in G \mid \phi(g) = e_H\}.$$

Prove that $\ker(\phi) = \{e_G\} \iff \phi$ is injective.

Exercise 8.8 Conjugacy (exam 2024) ($\star\star$)

This exercise is taken from the spring exam of 2024.
Consider a *finite* group $\langle G; \star, \widehat{\ }, e_G\rangle$. Let H be a subgroup of G. **Prove** that

$$T = \{g \in G \mid \forall h \in H,\ g \star h \star \widehat{g} \in H\}$$

is a subgroup of G.
Hint: use without proof that any injective function from a finite set to itself is also surjective.

Due by 13/11/2025, 23:59.
Exercise 8.4 will be graded.