Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

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Exercise sheet 12

This is the exercise sheet number 12. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . The graded exercise is Exercise 12.5 and your solution has to be uploaded on the Moodle page of the course **by 11/12/2025, 23:59**. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 12.1 CNF and DNF ($\star$)

1. Let $F = (\neg(A \to C)) \leftrightarrow (A \to B)$. Using the method of function tables demonstrated in the lecture and discussed in the lecture notes (cf. proof of theorem 6.4 and example 6.14), construct a formula in CNF that is equivalent to F and a formula in DNF that is equivalent to F.

2. Let $G = (A \wedge \neg B) \vee (\neg A \wedge (C \wedge D))$. Using the equivalences from Lemma 6.1, construct a formula in CNF that is equivalent to G. In each step write which equivalence you use.

Exercise 12.2 Free (PL-)Variables ($\star$)

Determine all occurrences of free (PL-)variables in the following formulas:

i) $\forall x \forall y \, (P(x,y) \vee P(x,z))$
ii) $\forall x \, (\exists x \, P(x) \wedge P(x)) \vee P(x)$
iii) $\forall x \, \big(\exists y \, P(y,x) \vee \exists z \, Q(x, f(z))\big)$

Exercise 12.3 Interpretations ($\star$)

1. Which of the interpretations **i)**, **ii)** and **iii)** are models for the following formula? Justify your answers. (The symbol | denotes the divisibility relation.)

$$F = \forall x \forall y \forall z \, \Big(P(f(x,y), x) \;\wedge\; P(f(x,y), y) \;\wedge\; \big(\neg P(x,y) \to \neg P(x, f(y,z))\big)\Big)$$

i) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}, \quad f^{\mathcal{A}}(x,y) = x \cdot y, \quad P^{\mathcal{A}}(x,y) = 1 \iff y \mid x$
ii) $U^{\mathcal{A}} = \mathbb{N} \setminus \{0\}, \quad f^{\mathcal{A}}(x,y) = x^y, \quad P^{\mathcal{A}}(x,y) = 1 \iff y \mid x$
iii) $U^{\mathcal{A}} = \mathcal{P}(\mathbb{N}), \quad f^{\mathcal{A}}(A,B) = A \cap B, \quad P^{\mathcal{A}}(A,B) = 1 \iff A \subseteq B$

2. Let $G = \big(\forall x \, \exists y \, P(x,y)\big) \;\wedge\; \big(\forall y \, \exists x \, P(x,y)\big) \;\wedge\; \big(\forall x \, \forall y \, (P(x,y) \to \neg P(y,x))\big)$. Find an interpretation with *a finite universe* that is

   i) not suitable for G.
   ii) suitable but not a model for G.
   iii) suitable and a model for G.

Exercise 12.4 Predicate Logic with Equality ($\star$)

We extend the syntax and the semantics of predicate logic as follows:
Syntax: If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula.
Semantics: If F is of the form $(t_1 = t_2)$ for terms t_1 and t_2, then $\mathcal{A}(F) = 1$ if and only if $\mathcal{A}(t_1) = \mathcal{A}(t_2)$.

1. Let $F = \forall x \forall y\ (x = y)$. Find the necessary and sufficient conditions for an interpretation $\mathcal{A}$ to be a model for F. Justify your answer.

2. Let $G = \exists x \exists y\ \neg(x = y)$. Find the necessary and sufficient conditions for an interpretation $\mathcal{A}$ to be a model for G. Justify your answer.

3. Find a formula with equality H, such that for any interpretation $\mathcal{A}$ suitable for H, we have $\mathcal{A}(H) = 1 \iff |U^{\mathcal{A}}| \geq 3$.

Exercise 12.5 Statements about Formulas ($\star\,\star$) — GRADED *(8 points)*
Please upload your solution by 11/12/2025

Prove or disprove each of the following statements. Do not use any theorems or lemmas from the lecture notes.
Expectation: Your proofs should have the same level of detail and formality as the proof of lemma 6.7 7) of the lecture notes. Also, if you decide to prove a logical equivalence by showing logical consequences on both directions, and if the second direction is **completely analogous** to the first one, you can only indicate it instead of writing out the (almost) identical proof again.

1. (3 points) For any formulas F and G, we have

$$\exists x(F \vee G) \equiv (\exists x F) \vee (\exists x G).$$

2. (3 points) For any formula F, we have

$$\exists x \exists y F \equiv \exists y \exists x F.$$

3. (2 points) For any formula F, we have

$$\forall x \exists y F \equiv \exists y \forall x F.$$

Exercise 12.6 An extension of propositional logic (exam FS 2025) ($\star$)

This exercise is taken from the spring exam of 2025.
Extend propositional logic with a symbol $\diamond$ as follows:

- **Syntax:** if F and G are formulas, then $F \diamond G$ is a formula.
- **Semantics:** $\mathcal{A}(F \diamond G) = 1 \iff \mathcal{A}(F) = 0$ or $\mathcal{A}(G) = 0$ (or both).

Prove or disprove that given any propositional logic formula F, there exists a formula G such that $G \equiv F$ and G contains only atomic formulas (cf. definition 6.23) and the symbol $\diamond$ (and parentheses).

Due by 11/12/2025, 23:59.
Exercise 12.5 will be graded.