Diskrete Mathematik HS2025 — Prof. Dennis HOFHEINZ

Marian DIETZ — Milan GONZALEZ-THAUVIN — Zoé REINKE

Exercise sheet 13

This is the exercise sheet number 13, last one of the semester. The difficulty of the questions and exercises are rated from very easy ($\star$) to hard ($\star\star\star\star$) . There is no bonus exercise this week. Because this is the last sheet, some exercises need content that will be covered next Monday and Wednesday.

Exercise 13.1   Warm-Up ($\star$)

1. Are the formulas $F = P(x) \wedge \neg Q(y)$ and $G = \neg Q(x) \wedge P(y)$ equivalent?

2. Find a formula H which has both x and y as free variables, such that $\forall x H \equiv \forall y H$.

3. What does it mean for a (propositional) calculus to be **complete**?

Exercise 13.2   Tautologies ($\star\star$)

Let F be a formula with free variables $x_1, ..., x_n$ (and no others). Show that F is valid if and only if $\forall x_1 \ldots \forall x_n F$ is valid.

Exercise 13.3   Prenex Normal Form ($\star$)

For each of the following formulas, find an equivalent formula in the prenex normal form.

1. $\big(\forall x\, P(x)\big) \to Q(x)$

2. $\forall z \exists y \, \big(P(x, g(y), z) \vee \neg \forall x Q(x)\big) \, \wedge \, \neg \forall z \exists x \, \neg R(f(x, z), z)$

Exercise 13.4   Formulas and Statements ($\star\star$)

For each of the following expressions, determine whether it is syntactically correct, and, if so, whether it is a formula or a statement about formulas[1]. If an expression is a statement, decide whether it is true or false (each time justify your answer).

1. $\forall x \, \exists y \, \big(P(z) \leftrightarrow Q(f(f(x, z), y))\big)$

2. $\big(\forall x\, P(x)\big) \models P(x)$

3. $(P(x) \models P(x)) \equiv Q(x)$

4. $\{P(x), P(f(a))\} \models P(a)$

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[1]Whenever parentheses are not necessary, they can be omitted. Parentheses do not influence correctness.

Exercise 13.5 The Barber of Zurich (⋆)

Use Corollary 6.13 to show that there does not exist a barber in Zurich who shaves all those and exactly those who do not shave themselves.

Exercise 13.6 Calculi

1. (⋆) Decide which of the following rules are correct (= sound) and justify your answers:

$$\{F\} \vdash_{R_1} F \vee G \qquad \{F \wedge G\} \vdash_{R_2} F \qquad \{\neg(F \wedge G)\} \vdash_{R_3} \neg F \wedge \neg G$$

$$\{F, F \to G\} \vdash_{R_4} G \qquad \{F \to G\} \vdash_{R_5} \neg F \to \neg G \qquad \{F, G\} \vdash_{R_6} F \wedge G$$

2. (⋆ ⋆) Let K be the calculus consisting of the **correct** rules of question 1 only. Using K, derive formally the formula $((A \wedge B) \wedge C) \wedge D$ from the following set of formulas:

$$\{(D \wedge A) \to C, \; A \wedge B, \; B \wedge A, \; (B \vee C) \to D\}$$

3. (⋆ ⋆) Is $K' = \{R_2, R_4\}$ complete? Justify your answer.

4. (⋆ ⋆) Give an example of a calculus, which is complete but not sound.

Exercise 13.7 Resolution

1. (⋆) Prove the following statements using the resolution calculus.

   i) $F = (A \vee B) \wedge (\neg E) \wedge (\neg B \vee D) \wedge (\neg D \vee E) \wedge (\neg A \vee B)$ is not satisfiable.
   ii) $G = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$ is a tautology.
   iii) $H = A \wedge C$ is a logical consequence of $M = \{A \to C, B \to A, A \vee B\}$.

2. (⋆ ⋆ ⋆) Let $\mathcal{K}$ be a finite set of finite clauses and let $\mathcal{K}_0, \mathcal{K}_1, \ldots$ be a sequence of applications of derivation rules, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\text{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$. Show that there exists an n such that $\mathcal{K}_m = \mathcal{K}_n$ for all $m > n$.
   Hint: Intuitively, the goal of the question is to show that from a finite set of finite clauses, after a finite number of applications of derivation rules, no new clauses can be derived.

3. (⋆ ⋆ ⋆) Show that the statement from question 2 is no longer true for an infinite set $\mathcal{K}$ of finite clauses. More precisely, let $\mathcal{K} = \{\{A_j, \neg A_{j+1}\} \mid j \in \mathbb{N}\}$. Show that there exists an infinite sequence $\mathcal{K}_0, \mathcal{K}_1, \ldots$, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\text{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$, and for all $i > 0$, $\mathcal{K}_i \neq \mathcal{K}_{i-1}$.

Exercise 13.8 A special calculus (exam FS 2025) (⋆ ⋆)

This exercise is taken from the spring exam of 2025.
Consider the calculus consisting of the following derivation rules:

$$\varnothing \vdash_{R_1} F \to (G \to F),$$
$$\varnothing \vdash_{R_2} (F \to (G \to H)) \to ((F \to G) \to (F \to H)),$$
$$\{F, F \to G\} \vdash_{R_3} G.$$

Formally derive $A \to C$ from $\{A \to B,\ B \to C\}$ in the calculus.

No exercise will be graded.