D-INFK
Bernd Gärtner
Robert Weismantel

Linear Algebra

HS 2025

Assignment 2

Submission Deadline: **7 October, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Rank of a matrix (in-class) (★★☆)

Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$.

a) Calculate A_m for $m \in \{2, 3, 4\}$.

b) Determine the rank of A. You need to motivate your answer.

2. Nullspace as a hyperplane (hand-in) (★★☆)

Let $\mathbf{v} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ not all zero. Consider the $m \times n$ matrix A of the form

$$A = \begin{bmatrix} | & | & & | \\ \lambda_1 \mathbf{v} & \lambda_2 \mathbf{v} & \cdots & \lambda_n \mathbf{v} \\ | & | & & | \end{bmatrix}$$

a) What is the rank of the matrix A?

b) Prove that the nullspace $\mathbf{N}(A)$ is a hyperplane through the origin.

3. Matrix transformations (★★☆)

a) Consider the matrix transformation given by the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Describe the geometric operation that this matrix transformation corresponds to.

b) Consider the line through the origin $L = \{\lambda \mathbf{v} : \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with $\mathbf{v} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^\top$. Find a matrix $A \in \mathbb{R}^{3 \times 3}$ that corresponds to rotating vectors by $180°$ around the axis L.

Hint: Try to find out what $A\mathbf{e}_1$, $A\mathbf{e}_2$, and $A\mathbf{e}_3$ should be.

4. Scalar product (★★☆)

Recall that the scalar product of two vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

in $\mathbb{R}^n$ is a real number given by

$$\mathbf{v}^\top \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

for the covector $\mathbf{v}^\top \in (\mathbb{R}^n)^*$. The vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal to each other if and only if $\mathbf{v}^\top \mathbf{w} = 0$.

Let $A \in \mathbb{R}^{m \times n}$ be the matrix

$$A = \begin{bmatrix} — & \mathbf{u}_1^\top & — \\ — & \mathbf{u}_2^\top & — \\ & \vdots & \\ — & \mathbf{u}_m^\top & — \end{bmatrix}$$

with rows $\mathbf{u}_1^\top, \mathbf{u}_2^\top, \ldots, \mathbf{u}_m^\top \in (\mathbb{R}^n)^*$. Recall that, by Observation 2.8, we have

$$A\mathbf{x} = \begin{bmatrix} — & \mathbf{u}_1^\top & — \\ — & \mathbf{u}_2^\top & — \\ & \vdots & \\ — & \mathbf{u}_m^\top & — \end{bmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{u}_1^\top \mathbf{x} \\ \mathbf{u}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{u}_m^\top \mathbf{x} \end{pmatrix}$$

for $\mathbf{x} \in \mathbb{R}^n$. In particular, we have $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x}$ is orthogonal to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

a) Now consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ and let $\lambda, \mu \in \mathbb{R}$ be arbitrary. Prove that the vector $\lambda\mathbf{x} + \mu\mathbf{y}$ is orthogonal to each of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

b) Finally, consider the set of vectors $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ and assume $|\mathcal{L}| \geq 2$. Is $\mathcal{L}$ a finite set?

5. Rank of matrices (★☆☆)

a) What is the rank of the following 2×3 matrix A?

$$A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & 6 & 0 \end{bmatrix}$$

b) What is the rank of the following 3×3 matrix A?

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

6. Skew-symmetric matrices (★★☆)

A square matrix $A \in \mathbb{R}^{m \times m}$ is skew-symmetric if and only if $A^\top = -A$.

a) Give an example of a nonzero skew-symmetric matrix when $m = 2$.

b) Let $A = [a_{ij}]_{i=1,j=1}^{m\ \ m}$ be skew-symmetric. Show that $a_{ii} = 0$ for all $i \in [m]$.

c) Which matrices in $\mathbb{R}^{m\times m}$ are both skew-symmetric and symmetric? Argue why your list is complete.

d) Let $A \in \mathbb{R}^{3\times 3}$ be skew-symmetric. Show that $\text{rank}(A) \leq 2$.

7. Embedding a line in $\mathbb{R}^m$ (★☆☆)

Let $\mathbf{v} \in \mathbb{R}^m$ be non-zero and consider the line $L = \{\lambda\mathbf{v} \,:\, \lambda \in \mathbb{R}\}$. Prove that there is a matrix transformation $T_A : \mathbb{R}^1 \to \mathbb{R}^m$ with $\{T_A(\mathbf{x}) \,:\, \mathbf{x} \in \mathbb{R}^1\} = L$.