D-INFK Linear Algebra HS 2025
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Assignment 9

Submission Deadline: **25 November, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA and bonus points for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Properties of pseudoinverses (in-class) (★★☆)

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ be arbitrary matrices.

a) Prove that if $\text{rank}(A) = \text{rank}(B) = n$, we have $(AB)^\dagger = B^\dagger A^\dagger$.

b) Prove that $A^\dagger A A^\dagger = A^\dagger$.

c) Prove that $(A^\top)^\dagger = (A^\dagger)^\top$.

2. Characterizing solvability via nullspaces (bonus, hand-in) (★★☆)

Let $A \in \mathbb{R}^{m\times n_1}$ and $B \in \mathbb{R}^{m\times n_2}$. Consider the system of linear equations

$$\text{(EQ)} \qquad Ax + By = c, \qquad c \in \mathbb{R}^m$$

where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ are the unknown vectors. Show that the system has a solution (x, y) for *every* right-hand side $c \in \mathbb{R}^m$ if and only if the null spaces of $A^\top$ and $B^\top$ satisfy

$$\mathbf{N}(A^\top) \cap \mathbf{N}(B^\top) = \{0\}$$

Hint: *Use Theorem 6.2.4. from the lecture notes.*

3. Parametric solvability example (★☆☆)

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Use the result from Exercise 2 to determine all values of the parameters $b_1, b_2, b_3 \in \mathbb{R}$ for which the system

$$\text{(EQ)} \qquad Ax + By = c$$

admits a solution (x, y) for *every* right-hand side $c \in \mathbb{R}^3$.

4. Characterization of equality in $\mathbf{C}(A)$ (★☆☆)

Let $A \in \mathbb{R}^{m\times n}$ a matrix and $x, y \in \mathbf{C}(A^\top)$. Show that

$$AA^\top x = AA^\top y \iff x = y.$$

5. Bijection between $\mathbf{C}(A^\top)$ and $\mathbf{C}(A)$ (★★☆)

Let $A \in \mathbb{R}^{m\times n}$ be an arbitrary matrix with column space $\mathbf{C}(A)$ and row space $\mathbf{C}(A^\top)$. Consider the function $f : \mathbf{C}(A^\top) \to \mathbf{C}(A)$ that maps $\mathbf{x} \in \mathbf{C}(A^\top)$ to $A\mathbf{x} \in \mathbf{C}(A)$. Prove that f is bijective.

6. Pseudoinverses and projection matrices (★★☆)

Let $A \in \mathbb{R}^{m\times n}$ be an arbitrary matrix. Prove that $A^\dagger A$ is symmetric and that it is the projection matrix for the subspace $\mathbf{C}(A^\top)$.