Bernd Gärtner
Robert Weismantel

Assignment 10

Submission Deadline: **02 December, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Gram-Schmidt (in-class) (★☆☆)

Consider the invertible matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

a) Apply the Gram-Schmidt process to the columns of A.

b) Write down a QR-decomposition of A.

c) Apply the Gram-Schmidt process to the columns of B.

d) Is it always true that the Gram-Schmidt process on the columns of an upper triangular $n \times n$ matrix with non-zero diagonal entries yields the canonical basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$? Provide a proof or counterexample.

2. Orthogonal 2×2 matrices and rotation matrices (hand-in) (★★☆)

Recall from Assignment 3 Exercise 6 or Example 6.3.4 that a 2×2 rotation matrix is defined to be

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

for some angle $\phi \in \mathbb{R}$.

a) Find an orthogonal 2×2 matrix that is not a rotation matrix.

b) Let $A \in \mathbb{R}^{2\times 2}$. Prove that $|\det(A)| = 1$ if A is orthogonal.

c) Prove that the converse is not true, i.e., find a matrix A that is not orthogonal but we still have $|\det(A)| = 1$.

3. Preserving inner products (★☆☆)

Let $Q \in \mathbb{R}^{m \times m}$ be an arbitrary matrix. Let $(Q\mathbf{v})^\top(Q\mathbf{w}) = \mathbf{v}^\top\mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$. Show that Q is orthogonal.

4. Permutation matrices (★★☆)

Let $P \in \mathbb{R}^{n \times n}$ be a permutation matrix for some $n \geq 1$. In particular, P has the form

$$P = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{e}_{p(1)} & \mathbf{e}_{p(2)} & \cdots & \mathbf{e}_{p(n)} \\ | & | & \cdots & | \end{bmatrix}$$

where $p : [n] \to [n]$ is a bijective function (the permutations of $[n]$ are exactly the bijective funtions $p : [n] \to [n]$). Prove that there exists $k \in \mathbb{N}$ with $P^k = I$.

5. Unique characterization of solution of linear system (★★★)

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and assume the system of linear equations

$$Ax = b, \qquad x \in \mathbb{R}^n$$

is unsolvable. Then, by Lemma 6.2.4 of the lecture notes we know that the system of linear equations

$$A^\top z = 0, \qquad b^\top z = 1, \qquad z \in \mathbb{R}^m$$

is solvable, i.e. there exists *at least* a solution $z \in \mathbb{R}^m$.

a) Show that there exists a *unique* $z \in C\big([A \;\; b]\big)$ satisfying

$$A^\top z = 0, \qquad b^\top z = 1$$

b) Show that this z can be written as

$$z = \frac{1}{q^\top q} q$$

where $b = p + q$ is the unique decomposition of b with $p \in C(A)$ and $q \in C(A)^\perp$.

6. Determinants of skew-symmetric matrices (★★★)

Recall that a square matrix $A \in \mathbb{R}^{n \times n}$ is called skew-symmetric if and only if $A = -A^\top$. Our goal is to prove that $\det(A) = 0$ if n is odd.

a) Let $B \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Show that $\det(\lambda B) = \lambda^n \det(B)$.

b) Let $A \in \mathbb{R}^{n \times n}$ be skew-symmetric. Show that $\det(A) = 0$ if n is odd. You are allowed to use Theorem 7.2.5.

c) Let n be even. Show that there exists a skew-symmetric $A \in \mathbb{R}^{n \times n}$ such that $\det(A) \neq 0$. Justify your answer.