Bernd Gärtner
Robert Weismantel

Assignment 11

Submission Deadline: **09 December, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Computing determinants (in-class) (★☆☆)

a) For what values of $a, b, c \in \mathbb{R}$ is the determinant of the following matrix zero? (You should justify your answer.)

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 & c \\ a & 5 & 0 & 4 & -1 \\ 2 & 1 & b & -1 & -3 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 3 & 1 \end{bmatrix}$$

Hint: Use Proposition 7.3.2.

b) It turns out that the determinant of a triangular matrix is easy to calculate (Proposition 7.2.4). Moreover, the determinant does not change when a multiple of a row is added to another row (and row swaps only change the sign). This allows us to efficiently determine the determinant of any matrix using Gauss elimination. Determine the determinant of

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

by performing the Gauss elimination manually.

2. Determinant of block matrix (hand-in) (★★☆)

a) Consider the four matrices

$$A \in \mathbb{R}^{m\times m}$$
$$C \in \mathbb{R}^{(n-m)\times(n-m)}$$
$$B \in \mathbb{R}^{m\times(n-m)}$$
$$0 \in \mathbb{R}^{(n-m)\times m}.$$

where $m, n \in \mathbb{N}^+$ with $n > m$. We can plug these matrices together as follows

$$M := \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

to obtain the $n \times n$ matrix M. Prove that $\det M = (\det A)(\det C)$.

Hint: Decompose $M = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ *and use use Definition 7.2.3 on both parts separately.*

b) Calculate the determinant of the following matrix

$$M = \begin{bmatrix} 2 & 0 & 0 & 4 & 0 & 7 \\ 9 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 5 & 0 & 7 \\ 2 & 3 & 1 & 5 & 0 & 2 \\ 8 & 8 & 7 & 3 & 2 & 1 \end{bmatrix}$$

by hand without using Gauss elimination.

***Hint**: Put M into the correct form and use the result from the previous subtask. You might need to swap some columns or rows and analyze how this affects the determinant.*

3. Complex numbers (★☆☆)

Given the complex numbers

$$u = 3 - i^3,$$
$$v = 1 + i,$$
$$w = 3 - 4i,$$

calculate the expressions $u + v + w,\ u \cdot v,\ v \cdot w \cdot i,\ {}^{w}/_{v},\ {}^{v}/_{u},\ |\,v\,|$.

4. Determinants of four matrices (★★☆)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n$ and $M \in \mathbb{R}^{(n-2)\times n}$ be arbitrary and consider the four $n \times n$ matrices

$$A = \begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & \mathbf{u}_1^\top & - \\ & M & \end{bmatrix}, B = \begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & \mathbf{u}_2^\top & - \\ & M & \end{bmatrix}, C = \begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & \mathbf{u}_1^\top & - \\ & M & \end{bmatrix}, D = \begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & \mathbf{u}_2^\top & - \\ & M & \end{bmatrix}$$

as well as the following $n \times n$ matrix

$$E = \begin{bmatrix} - & (\mathbf{v}_1 - \mathbf{v}_2)^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix}.$$

Find a formula for $\det(E)$ in terms of $\det(A), \det(B), \det(C), \det(D)$.

5. Eigenvalues when adding cI to matrices (★★☆)

a) Let $M \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$. Show that for each real eigenvalue $\lambda \in \mathbb{R}$ of M, $\lambda + c$ is a real eigenvalue of $M + cI$.

b) Using the property from **a)**, find two distinct real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ of the matrix

$$A = \begin{bmatrix} 3 & 3 & 5 & 7 & 9 & 11 \\ 1 & 5 & 5 & 7 & 9 & 11 \\ 1 & 3 & 7 & 7 & 9 & 11 \\ 1 & 3 & 5 & 9 & 9 & 11 \\ 1 & 3 & 5 & 7 & 11 & 11 \\ 1 & 3 & 5 & 7 & 9 & 13 \end{bmatrix}.$$

6. Composition of permutations (★☆☆)

Let $p, q : [n] \to [n]$ be two permutations of the set $\{1, \dots, n\}$, and $P, Q \in \mathbb{R}^{n \times n}$ be the associated (row-based) permutation matrices. In particular,

$$P_{ij} = \begin{cases} 1 & \text{if } p(i) = j \\ 0 & \text{otherwise} \end{cases}$$

Show that the matrix $PQ \in \mathbb{R}^{n \times n}$ is the associated permutation matrix of the permutation $q \circ p : [n] \to [n]$, defined as

$$(q \circ p)(i) = q\big(p(i)\big)$$