Mock Exam Solution

▼ 1a

$x = \frac{4}{3}$.

Explanation:

For the matrix product MN, we get

$$MN = \begin{bmatrix} 4+3x & 8+5x \\ 11 & 19 \end{bmatrix}.$$

We also compute

$$NM = \begin{bmatrix} 8 & 2x+12 \\ 11 & 3x+15 \end{bmatrix}.$$

The condition $MN = NM$ implies $4 + 3x = 8$. Solving for x we get $x = \frac{4}{3}$. We can check that with this choice of x, we indeed get $MN = NM$.

▼ 1b

Observe that $v_3 - w_3 = c$ and hence $c = 8$. By adding the two vector equations, we therefore get

$$2\mathbf{v} = \begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix}$$

and hence

$$\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}.$$

Plugging this into the equations again, we also conclude

$$\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

▼ 1c

> **Explanation**: Observe that we are given the linear system
>
> $$\begin{bmatrix} \vdots & \vdots \\ \mathbf{b}_1 & \mathbf{b}_2 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{v}$$
>
> that we want to solve for c_1 and c_2. Using our favourite method to solve linear systems (e.g. elimination), we obtain $c_1 = 1$ and $c_3 = 3$. Indeed, we can easily check that $\mathbf{b}_1 + 3\mathbf{b}_2 = \mathbf{v}$.

▼ 2a

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}.$$

Explanation: This can be read of from the definition of f. Indeed, in order to obtain $x_1 - x_2$ in the first entry, we need to appropriately use the coefficient 1 and -1 in the first row of A. Similarly, the second entry corresponds to the second row of A, and the third entry to the third row of A.

▼ 2b

Notice: $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

Computing $A + B = I \Rightarrow (A+B)^3 = I^3 = I$

Hence the solution is I.

▼ 3a

By definition of the length (2-norm) of a vector, we get

$$\begin{aligned} \|\mathbf{v}+\mathbf{w}\|^2 + \|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}+\mathbf{w})^\top(\mathbf{v}+\mathbf{w}) + (\mathbf{v}-\mathbf{w})^\top(\mathbf{v}-\mathbf{w}) \\ &= \left(\mathbf{v}^\top\mathbf{v} + 2\mathbf{v}^\top\mathbf{w} + \mathbf{w}^\top\mathbf{w}\right) + \left(\mathbf{v}^\top\mathbf{v} - 2\mathbf{v}^\top\mathbf{w} + \mathbf{w}^\top\mathbf{w}\right) \\ &= 2\mathbf{v}^\top\mathbf{v} + 2\mathbf{w}^\top\mathbf{w} \\ &= 2\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right). \end{aligned}$$

▼ 3b

We prove this by direct calculation

$$\mathbf{v}\cdot\mathbf{w} = \sum_{i=1}^{n} v_i w_i \geq \sum_{i=1} \min\{v_i, w_i\}^2 = \sum_{i=1} u_i^2 = \|\mathbf{u}\|^2$$

where we crucially used that $v_i, w_i \geq 0$ to substitute both with $\min\{v_i, w_i\}$.

▼ 3c

We are given that $C(B) \subseteq N(A)$.

This means that every column of B lies in the nullspace of A.

$\Rightarrow AB = 0$

Apply the rank-nullity theorem to A.

As $A \in \mathbb{R}^{3\times 4}$, we have:

$rank(A) + dim(N(A)) = 4$

$\Rightarrow dim(N(A)) = 4 - rank(A)$

Using the constraint $C(B) \subseteq N(A)$, we have:

$rank(B) = dim(C(B)) \leq dim(N(A))$

$\Rightarrow$ From Steps 1 and 2:

$rank(B) \leq dim(N(A)) = 4 - rank(A)$

$\Rightarrow rank(A) + rank(B) \leq 4$

▼ 4a

Proof by contradiction.

Assume $I - B$ is not invertible.

$\Rightarrow$ there exists a nonzero vector $x \in \mathbb{R}^m$ such that $(I - B)x = 0$, i.e. $Bx = x$ with $x \neq 0$.

$\Rightarrow$ Let i be an index where $|x_i|$ is maximal among all components, i.e., $|x_i| \geq |x_j| \quad \forall j = 1, 2, ..., m$ and $|x_i| > 0$ (since x is nonzero).

$\Rightarrow$ From $Bx = x$, the i-th component gives: $x_i = \sum_{j=1}^{m} b_{ij} \cdot x_j$

$\Rightarrow |x_i| = |\sum_{j=1}^{m} b_{ij} \cdot x_j| \leq \sum_{j=1}^{m} |b_{ij}| \cdot |x_j|$

$\Rightarrow |x_i| \leq \sum_{j=1}^{m} |b_{ij}| \cdot |x_j| \leq |x_i| \cdot \sum_{j=1}^{m} |b_{ij}| = |x_i| \cdot s_i$ (As $|x_j| \leq |x_i|$)

$\Rightarrow$ As $|x_i| > 0$, we can divide both sides by $|x_i|$: $1 \leq s_i$

$\Rightarrow$ Contradiction as s_i should be less that 1, hence $I - B$ is invertible.

▼ 4b

b) No, T is not a linear functional. Consider the standard unit vector $\mathbf{e}_n \in \mathbb{R}^n$ and choose $\lambda = 2$. By definition of T, we get

$$T(\lambda \mathbf{e}_n) = \sum_{k=1}^{n-1} \lambda^k 0 + \lambda^n = \lambda^n$$

and also

$$T(\mathbf{e}_n) = \sum_{k=1}^{n-1} 0 + 1 = 1.$$

Hence, we have $T(\lambda \mathbf{e}_n) = \lambda^n = 2^n \neq 2 = \lambda T(\mathbf{e}_n)$ which means that T is not a linear functional. Note that the assumption $n \geq 2$ is crucial for this last step to work.

▼ 4c

We define the vectors

$$\mathbf{a} = \begin{bmatrix} \frac{a_1}{\sqrt{b_1}} \\ \frac{a_2}{\sqrt{b_2}} \\ \vdots \\ \frac{a_n}{\sqrt{b_n}} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \sqrt{b_1} \\ \sqrt{b_2} \\ \vdots \\ \sqrt{b_n} \end{bmatrix}$$

and obtain

$$\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \geq (\mathbf{a} \cdot \mathbf{b})^2$$

by using Cauchy-Schwarz (and squaring both sides). Now observe that

$$\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 = \left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \right) (b_1 + b_2 + \cdots + b_n)$$

and

$$(\mathbf{a} \cdot \mathbf{b})^2 = \left(\frac{a_1}{\sqrt{b_1}} \sqrt{b_1} + \frac{a_2}{\sqrt{b_2}} \sqrt{b_2} + \cdots + \frac{a_n}{\sqrt{b_n}} \sqrt{b_n} \right)^2 = (a_1 + a_2 + \cdots + a_n)^2 .$$

Plugging everything together, we conclude

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i^2}{b_i} \right) \left(\sum_{i=1}^{n} b_i \right)$$

which can be rearranged to

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

▼ 5.1

c)

Explanation: Let $\mathbf{w} \in \mathbb{R}^3$ be a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Any A with $\mathbf{R}(A) = \operatorname{Span}(\mathbf{w})$ satisfies $A\mathbf{u} = A\mathbf{v} = \mathbf{0}$ but does not have rank 0 , hence a) is not correct. Option $\mathbf{b}$) is not correct either: $A = 0$ is a counterexample.

It remains to observe that option d) is false and therefore option c) is correct. To see this, observe that $A(\mathbf{u} - \mathbf{v}) = \mathbf{0}$ and that by assumption we must have $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$. This implies that A must have a non-trivial nullspace, i.e. $\dim(\mathbf{N}(A)) \geq 1$.

We conclude $\operatorname{rank}(A) = \dim(\mathbf{R}(A)) = 3 - \dim(\mathbf{N}(A)) \leq 2$.

▼ 5.2

c)

- a)❌
  The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has $\mathbf{N}(A) \neq \{0\}$ and is not nilpotent since $A^k = A$ for all $k \in \mathbb{N}^+$.
- b)❌
  The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is nilpotent. The matrix $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is nilpotent too. But $A + B$ has full rank and hence cannot be nilpotent.
- c)✅
  This is true: a full rank matrix cannot be nilpotent since the product of square full rank matrices is square and full rank again. Hence, a matrix that is nilpotent cannot have full rank.
- d)❌
  Consider again $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. They are both nilpotent but $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not.

▼ 5.3

b)

For something to be a linear transformation:

1. $f(x + y) = f(x) + f(y)$ (additivity)
2. $f(cx) = cf(x)$ (homogeneity)

Thus for homogeneity**:**

$f(cx) = -2v^\top(cx) + \|v\|^2 = -2cv\top x + \|v\|^2$

$cf(x) = c(-2v^\top x + \|v\|^2) = -2cv^\top x + c\|v\|^2$

For f to be linear, we need $f(cx) = cf(x)$, which requires: $\|v\|^2 = c\|v\|^2 \quad \forall c$

$\Rightarrow v = 0$

▼ 5.4

c)

(a) Given arbitrary $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have $f(\mathbf{v} + \alpha\mathbf{w}) = 2A(\mathbf{v} + \alpha\mathbf{w}) = 2A\mathbf{v} + \alpha 2A\mathbf{w} = f(\mathbf{v}) + \alpha f(\mathbf{w})$ which proves that f is linear.

(b) Given arbitrary $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have $f(\mathbf{v} + \alpha\mathbf{w}) = A^2(\mathbf{v} + \alpha\mathbf{w}) = A^2\mathbf{v} + \alpha A^2\mathbf{w} = f(\mathbf{v}) + \alpha f(\mathbf{w})$ which proves that f is linear.

(c) This function is not linear since $f(2\mathbf{v}) = (A2\mathbf{v})^\top(A2\mathbf{v}) = 4(A\mathbf{v})^\top(A\mathbf{v}) = 4f(\mathbf{v}) \neq 2f(\mathbf{v})$ in general (the only case where this breaks down is if A is the zero matrix).

(d) Given arbitrary $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have $f(\mathbf{v} + \alpha\mathbf{w}) = \mathbf{1}^\top A(\mathbf{v} + \alpha\mathbf{w}) = \mathbf{1}^\top A\mathbf{v} + \alpha\mathbf{1}^\top A\mathbf{w} = f(\mathbf{v}) + \alpha f(\mathbf{w})$ which proves that f is linear.

▼ 5.5

b)

$$A^2 = 3A - I$$

$$\Rightarrow A^2 - 3A + I = 0$$

$$\Rightarrow A - 3I + A^{-1} = 0 \text{ (Multiply } A^{-1}\text{)}$$

$$\Rightarrow A^{-1} = 3I - A$$

$$\Rightarrow A + A^{-1} = A + (3I - A) = 3I$$

▼ 5.6

c)

Explanation: Clearly, the two vectors in a) are linearly independent as they are a subset of the original four vectors.

The three vectors in b) are linearly independent as well: One can e.g. see this by observing that all three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be obtained by the given vectors. For example, $\mathbf{v}_1$ can be obtained as $\mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) - \frac{1}{2}(\mathbf{v}_2 + \mathbf{v}_3) + \frac{1}{2}(\mathbf{v}_3 + \mathbf{v}_1)$.

However, this does not work in c). Indeed, the four vectors in c) are linearly dependent as evidenced by the linear combination

$$(\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_2 + \mathbf{v}_3) + (\mathbf{v}_3 + \mathbf{v}_4) - (\mathbf{v}_4 + \mathbf{v}_1) = \mathbf{0}.$$

Finally, the three vectors in option d) are linearly independent again as all three vectors $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ are spanned by the three given vectors. We conclude that option c) is correct.