Projections

Valentin Gröner

1 Definition

Given a subspace $S \subseteq \mathbb{R}^m$ and a vector $\mathbf{b} \in \mathbb{R}^m$, we want to find the *projection* of $\mathbf{b}$ onto S. In $\mathbb{R}^2$ and $\mathbb{R}^3$, we have a clear intuition what this projection should be. This is illustrated in the following figures.

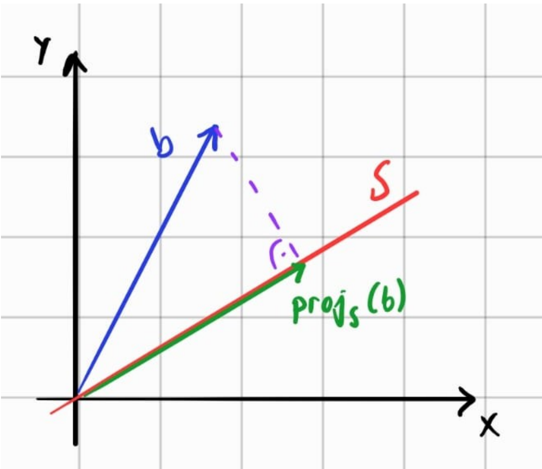

Figure 1: Projection of a vector in $\mathbb{R}^2$.

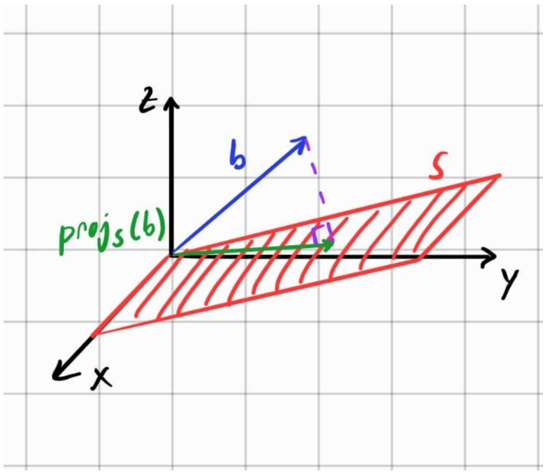

Figure 2: Projection of a vector in $\mathbb{R}^3$.

For general $\mathbb{R}^m$, we have to *define* the concept of a projection, since it is not geometrically clear what it should be. One approach is the following definition.

Definition 1.1 (Projection). *For a subspace $S \subseteq \mathbb{R}^m$ and a vector $\mathbf{b} \in \mathbb{R}^m$, we define its projection as the vector*

$$\text{proj}_S(\mathbf{b}) := \operatorname*{argmin}_{\mathbf{p} \in S} \|\mathbf{b} - \mathbf{p}\|.$$

This corresponds to the vector in S with minimal distance to $\mathbf{b}$. We call the vector $\mathbf{e} := \mathbf{b} - \text{proj}_S(\mathbf{b}) \in \mathbb{R}^m$ the error vector.

2 Existence and Uniqueness

We now want to prove that Definition 1.1 always yields a unique vector $\mathbf{p} \in S$. As we have seen, $\mathbf{b}$ can be written in terms of the error vector and the projection as

$$\mathbf{b} = \text{proj}_S(\mathbf{b}) + \mathbf{e}.$$

From our intuition in 2 and 3 dimensions, we might suspect that $\mathbf{e}$ should be orthogonal to S to minimize distance to $\mathbf{b}$. More formally $\mathbf{e} \in S^\perp$. We know from the orthogonal complement that this decomposition of a vector into a part in S, $\text{proj}_S(\mathbf{b})$, and one in $S^\perp$, $\mathbf{e}$, is unique. We now prove that this decomposition of $\mathbf{b}$ actually corresponds to the decomposition in projection and error vector.

Theorem 2.1. *Let $S \subseteq \mathbb{R}^m$ be a subspace and $\mathbf{b} \in \mathbb{R}^m$ arbitrary. For the unique vectors $\mathbf{p} \in S, \mathbf{e} \in S^\perp$ that satisfy $\mathbf{b} = \mathbf{p} + \mathbf{e}$, it holds that $\mathbf{p} = \mathrm{proj}_S(\mathbf{b})$.*

Proof. There exist unique vectors $\mathbf{p}, \mathbf{e} \in \mathbb{R}^m$ such that

$$\mathbf{b} = \mathbf{p} + \mathbf{e}, \qquad \mathbf{p} \in S,\ \mathbf{e} \in S^\perp.$$

Let $\mathbf{p}' \in S$ be arbitrary. Then

$$\begin{aligned}
\|\mathbf{b} - \mathbf{p}'\|^2 &= \|\mathbf{b} - \mathbf{p} + \mathbf{p} - \mathbf{p}'\|^2 && (1)\\
&= \|\mathbf{e} + \mathbf{p} - \mathbf{p}'\|^2 && (2)\\
&= (\mathbf{e} + \mathbf{p} - \mathbf{p}')^\top (\mathbf{e} + \mathbf{p} - \mathbf{p}') && (3)\\
&= \mathbf{e}^\top \mathbf{e} + 2(\mathbf{p} - \mathbf{p}')^\top \mathbf{e} + (\mathbf{p} - \mathbf{p}')^\top (\mathbf{p} - \mathbf{p}'). && (4)
\end{aligned}$$

Since $\mathbf{p}, \mathbf{p}' \in S$ we have $\mathbf{p} - \mathbf{p}' \in S$, hence $\mathbf{e} \perp (\mathbf{p} - \mathbf{p}')$ and so

$$\|\mathbf{b} - \mathbf{p}'\|^2 = \|\mathbf{e}\|^2 + \|\mathbf{p} - \mathbf{p}'\|^2 \geq \|\mathbf{e}\|^2 = \|\mathbf{b} - \mathbf{p}\|^2,$$

with equality iff $\mathbf{p}' = \mathbf{p}$. Therefore, $\mathbf{p}$ minimizes the distance and $\mathbf{p} = \mathrm{proj}_S(\mathbf{b})$. □

Corollary 2.1 (Existence and Uniqueness)**.** *For any subspace $S \subseteq \mathbb{R}^m$ and any vector $\mathbf{b} \in \mathbb{R}^m$, $\mathrm{proj}_S(\mathbf{b})$ exists and is* unique.

Corollary 2.2 (Orthogonality)**.** *For any subspace $S \subseteq \mathbb{R}^m$ and any vector $\mathbf{b} \in \mathbb{R}^m$, the difference vector between $\mathbf{b}$ and its projection onto S is orthogonal to S. In formulas,*

$$\mathbf{e} = (\mathrm{proj}_S(\mathbf{b}) - \mathbf{b}) \in S^\perp.$$

Corollary 2.3. *For any subspace $S \subseteq \mathbb{R}^m$ and any vector $\mathbf{b} \in \mathbb{R}^m$,*

$$\mathbf{b} = \mathrm{proj}_S(\mathbf{b}) + \mathrm{proj}_{S^\perp}(\mathbf{b}).$$

Proof. Let $\mathbf{b} \in \mathbb{R}^n$ be arb. We know there is a unique decomposition

$$\mathbf{b} = \mathbf{p} + \mathbf{e}, \quad \mathbf{p} \in S, \mathbf{e} \in S^\perp,$$

In the proof of Theorem 2.1 we saw that $\mathbf{p} = \mathrm{proj}_S(\mathbf{b})$, i.e. the part in S is the projection of $\mathbf{b}$ onto S. By applying the same argument to $S^\perp$, the part in $S^\perp$ is the projection of $\mathbf{b}$ onto $S^\perp$, i.e. $\mathbf{e} = \mathrm{proj}_S(\mathbf{b})$. □

Therefore, when we decompose a vector $\mathbf{b}$ into components in S and $S^\perp$, those components are exactly the projections of $\mathbf{b}$ onto the respective subspaces.

3 Computation

Now that we have properly defined projections to subspaces and gathered some insight into their properties, we can think about how to compute them.

Theorem 3.1. *Let $S \subseteq \mathbb{R}^m$ be a subspace with basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and let $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. Let $A \in \mathbb{R}^{m \times n}$ be the matrix whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then there exists a unique vector of coefficients $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that*

$$A\hat{\mathbf{x}} = \mathrm{proj}_S(\mathbf{b}),$$

and these coefficients satisfy the normal equations

$$A^\top A\hat{\mathbf{x}} = A^\top \mathbf{b}.$$

Proof. Since $\mathbf{e} = \mathbf{b} - \mathrm{proj}_S(\mathbf{b}) \in S^\perp$, we have for each basis vector $\mathbf{a}_i$ that $\mathbf{a}_i^\top \mathbf{e} = 0$, i.e.

$$\mathbf{a}_i^\top(\mathbf{b} - \mathrm{proj}_S(\mathbf{b})) = 0 \quad (i = 1, \ldots, n).$$

Writing these conditions in matrix form gives

$$A^\top(\mathbf{b} - \mathrm{proj}_S(\mathbf{b})) = \mathbf{0} \quad \Longrightarrow \quad A^\top\mathbf{b} = A^\top\mathrm{proj}_S(\mathbf{b}).$$

Because $\mathrm{proj}_S(\mathbf{b}) \in S$, there is a unique coefficient vector $\hat{\mathbf{x}}$ with $A\hat{\mathbf{x}} = \mathrm{proj}_S(\mathbf{b})$. Substituting yields the normal equations $A^\top A\hat{\mathbf{x}} = A^\top\mathbf{b}$. □

In the ideal case we can find a matrix P (the projection matrix) whose linear transformation maps every vector to its projection onto S, i.e.

$$\forall \mathbf{b} \in \mathbb{R}^m : \quad P\mathbf{b} = \mathrm{proj}_S(\mathbf{b}).$$

If $A^\top A$ is invertible, the normal equations give

$$\mathrm{proj}_S(\mathbf{b}) = A\hat{\mathbf{x}} = A(A^\top A)^{-1}A^\top\mathbf{b}.$$

Theorem 3.2. *$A^\top A$ is invertible if and only if the columns of A are linearly independent.*

Proof. Try the proof yourself or refer to the proof in the lecture notes. □

Since the columns of A form a basis of S, they are linearly independent and hence $A^\top A$ is invertible. We can thus define the projection matrix.

Definition 3.1 (Projection Matrix)**.** *For a subspace $S \subseteq \mathbb{R}^m$ with basis given by the columns of $A \in \mathbb{R}^{m\times n}$ (so that $A^\top A$ is invertible), the projection matrix $P \in \mathbb{R}^{m\times m}$ is*

$$P := A(A^\top A)^{-1}A^\top,$$

and it satisfies

$$\forall \mathbf{b} \in \mathbb{R}^m : \quad P\mathbf{b} = \mathrm{proj}_S(\mathbf{b}).$$